

Magnetic properties, Lyapunov exponent and superstability of the spin- $\frac{1}{2}$ Ising-Heisenberg model on diamond chain

N.S. Ananikian^{1,2} and V.V. Hovhannisyan¹

¹*A.I. Alikhanyan National Science Laboratory, 0036 Yerevan, Armenia. and*

²*Applied Mathematics Research Centre, Coventry University,
Coventry, CV1 5FB, England, United Kingdom.*

Abstract

The exactly solvable spin- $\frac{1}{2}$ Ising-Heisenberg model on diamond chain has been considered. We have found the exact results for the magnetization by using recursion relation method. The existence of the magnetization plateau has been observed at one third of the saturation magnetization in the antiferromagnetic case. Some ground-state properties of the model are examined. At low temperatures, the system has two ferrimagnetic (FRI1 and FRI2) phases and one paramagnetic (PRM) phase. Lyapunov exponents for the various values of the exchange parameters and temperatures have been analyzed. It has also been shown that the maximal Lyapunov exponent exhibits plateau. Lyapunov exponents exhibit different behavior for two ferromagnetic phases. We have found the existence of the supercritical point for the multi-dimensional rational mapping of the spin- $\frac{1}{2}$ Ising-Heisenberg model on diamond chain for the first time at absence of the external magnetic field and $T \rightarrow 0$ in the antiferromagnetic case.

I. INTRODUCTION

The investigation of physical properties of the low-dimensional quantum spin systems with competing interactions in an external magnetic field has been a subject of increasingly intense research interest in the recent decades. The research interest of these systems has attracted much attention due to the following reasons: first, they can be solved exactly by using different mathematical techniques, second, they are realized in the nature, and third, these systems present rich thermodynamic behavior, such as the appearance of magnetization plateaus, double peaks structure of the specific heat and magnetic susceptibility.

One of the interesting low-dimensional quantum spin system is the frustrated diamond Heisenberg spin-chain. The physical properties of real materials such as copper mineral $\text{Cu}_3(\text{CO}_3)_2(\text{OH})_2$, known as natural azurite (Copper Carbonate Hydroxide) can be well described by using the quantum antiferromagnetic Heisenberg model on a generalized diamond chain. The physical properties of the Heisenberg model on diamond chain have been investigated using different methods they are full numerical diagonalization and the Lanczos algorithm [1–3], the decoration-iteration transformation [4–7], the mapping transformation technique [8], the density-matrix renormalization-group (DMRG) and transfer-matrix renormalization-group (TMRG) techniques [9], Gibbs-Bogoliubov approach [10], cluster approach [11], the generalized gradient approximations (GGA) [12], the density functional theory and state-of-the-art numerical many body calculations [13].

Intriguing properties of the azurite made it a good candidate for studying its properties on the low-dimensional quantum spin systems. Kikuchi and co-workers [14] have experimentally studied the physical properties of the compound $\text{Cu}_3(\text{CO}_3)_2(\text{OH})_2$. They have shown that azurite can be regarded as a model substance of a distorted diamond chain. The temperature dependence on the magnetic susceptibility and specific heat shows double peak structure (around 20 and 5 K) on both magnetic susceptibility and specific heat results. The existence of the magnetization plateau at one third of the saturation magnetization has also been experimentally observed in the magnetization curve.

The aim of the recursion relation method is to cut lattice into branches and express the partition function of all lattice through the partition function of branches. This procedure will allow to derive one- or multi-dimensional mapping for branches of the partition function. After which the thermodynamic quantities of the physical system such as magnetization, magnetic susceptibility, specific heat can be expressed through recursion relation. One and multi dimensional mapping allows to investigate properties of different models for example Ising model on Husimi lattice [15, 16], zigzag ladder [17], triangular lattice [18], two-layer Bethe lattice [19, 20], mixed-spin Ising model on a decorated Bethe lattice [21, 22], Q-state Potts model on the Bethe lattice [16], zigzag ladder [23], phase diagrams for both ferromagnetic and antiferromagnetic cases, multicritical points, for the spin-1 Ising model on the Bethe lattice [24–27].

In this paper we have investigated some properties of the spin- $\frac{1}{2}$ Ising-Heisenberg model on diamond chain by using dynamical system (recursion relation) approach. Especially we have investigated magnetic properties of the model and shown the existence of the magnetization plateau at one third of the saturation value. The investigation of the ground-state properties of the model in the $\Delta - h$ plane shows the existence of three phases in the antiferromagnetic case and two phases in the ferromagnetic case. Another interesting property of the model has been found by investigating the behavior of Lyapunov exponent. Especially we have shown the existence of the plateau in the maximal Lyapunov exponent curve.

The rest of the paper is organized as follows: In the next section using the recursion relation method we derive the exact two dimensional recursion relations for the partition function of the spin- $\frac{1}{2}$ Ising-Heisenberg model on diamond chain. The exact results for the magnetization of Ising and Heisenberg spin sublattices have been derived. We describe the ground-state properties of the model in $\Delta - h$ plane. In Sec. III we have discussed the behavior of Lyapunov exponent. For the antiferromagnetic case the maximal Lyapunov exponent for the multi-dimensional rational mapping is considered and it is shown that near the magnetization plateaus the maximal Lyapunov exponent also exhibits plateau structure. The supercritical point at $h = 0$ and $T \rightarrow 0$ has been found. Finally, section IV contains concluding remarks.

II. RECURSION RELATION FOR THE ISING-HEISENBERG DIAMOND CHAIN

Let us consider the spin- $\frac{1}{2}$ Ising-Heisenberg model on diamond chain with free boundary conditions in the presence of an external magnetic field. The Hamiltonian operator of the model is equal to the summation of the plaquette

Hamiltonians and can be written as

$$\mathcal{H} = \sum_{i=1}^N \mathcal{H}_i = \sum_{i=1}^N [J(S_{a,i}^x S_{b,i}^x + S_{a,i}^y S_{b,i}^y + \Delta S_{a,i}^z S_{b,i}^z) + J_1 (S_{a,i}^z + S_{b,i}^z) (\mu_i^z + \mu_{i+1}^z) - h_H (S_{a,i}^z + S_{b,i}^z) - \frac{h_I}{2} (\mu_i^z + \mu_{i+1}^z)], \quad (1)$$

where \mathcal{H}_i is Hamiltonian of each plaquette, $S_{a,i}^\alpha, S_{b,i}^\alpha$ ($\alpha = x, y, z$) and μ_i^z represent relevant components of Heisenberg spin- $\frac{1}{2}$ and Ising spin- $\frac{1}{2}$ operators, the parameters J and J_1 stand for the interaction between the nearest-neighbouring Heisenberg pairs and the nearest-neighbouring Ising and Heisenberg spins, respectively and Δ is the anisotropy parameter. Hamiltonian (1) also includes longitudinal external magnetic fields h_H and h_I interacting with Heisenberg and Ising spins. The first summation in Eq. (1) is corresponding to the interstitial anisotropic Heisenberg spins coupling (J and Δ), the second summation is corresponding to the interaction between the nearest Ising and Heisenberg spins and the last two summations are corresponding to the field interaction with Ising and Heisenberg spins. In our further calculations we will consider the case when external magnetic field is uniform $h_H = h_I$. It is important to mention the separable nature of the Ising-type exchange interactions between neighboring Heisenberg dimers which are caused from the following commutation rule between different plaquette Hamiltonians: $[\mathcal{H}_i, \mathcal{H}_j] = 0$ for $i \neq j$.

The partition function of the system with Hamiltonian (1) is

$$Z = \sum_{\{\mu_i, S_{a,i}, S_{b,i}\}} \exp\{-\beta \mathcal{H}\}, \quad (2)$$

where $\beta = (k_B T)^{-1}$, k_B is Boltzmann constant (hereafter we consider $k_B = 1$) and T is the absolute temperature. By cutting diamond chain at $S_{a,0}$ and $S_{b,0}$ points (central plaquette) into two branches (we denote these branches $g_n(S_{a,0}, S_{b,0})$ see Fig. 1) the exact recursion relation for the partition function can be derived. After this procedure the partition function can be written as

$$Z = \sum_{\{S_{a,0}, S_{b,0}\}} e^{-\beta[J(S_{a,0}^x S_{b,0}^x + S_{a,0}^y S_{b,0}^y + \Delta S_{a,0}^z S_{b,0}^z) - h(S_{a,0}^z + S_{b,0}^z)]} g_n^2(S_{a,0}, S_{b,0}), \quad (3)$$

where $g_n^2(S_{a,0}, S_{b,0})$ is contribution of both left and right branches. The sum in Eq. (3) goes over all possible combinations of Heisenberg spins $S_{a,0}$ and $S_{b,0}$. Putting into Eq. (3) eigenvalues of the operator $\exp\{-\beta[J(S_{a,0}^x S_{b,0}^x + S_{a,0}^y S_{b,0}^y + \Delta S_{a,0}^z S_{b,0}^z) - h(S_{a,0}^z + S_{b,0}^z)]\}$ we can get the partition function expressed through $g_n(S_{a,0}, S_{b,0})$

$$Z = e^{-\frac{J\Delta}{4T} + \frac{h}{T}} g_n^2(\uparrow\uparrow) + e^{-\frac{J}{2T} + \frac{J\Delta}{4T}} g_n^2\left(\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}\right) + e^{\frac{J}{2T} + \frac{J\Delta}{4T}} g_n^2\left(\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}\right) + e^{-\frac{J\Delta}{4T} - \frac{h}{T}} g_n^2(\downarrow\downarrow), \quad (4)$$

where by \uparrow (up) and \downarrow (down) we denote directions of Heisenberg spins. To find recursion relations for the model we need to find relations between $g_n(S_{a,0}, S_{b,0})$ and $g_{n-1}(S_{a,1}, S_{b,1})$.

$$g_n(S_{a,0}, S_{b,0}) = \sum_{\{\mu_1, S_{a,1}, S_{b,1}\}} e^{-\beta[J(S_{a,1}^x S_{b,1}^x + S_{a,1}^y S_{b,1}^y + \Delta S_{a,1}^z S_{b,1}^z) + J_1(S_{a,0}^z + S_{b,0}^z)\mu_1^z + J_1\mu_1^z(S_{a,1}^z + S_{b,1}^z) - h(S_{a,1}^z + S_{b,1}^z + \mu_1^z)]} \quad (5) \\ * g_{n-1}(S_{a,1}, S_{b,1}).$$

Inserting into Eq. (5) eigenvalues of the operator $\exp\{-\beta[J(S_{a,1}^x S_{b,1}^x + S_{a,1}^y S_{b,1}^y + \Delta S_{a,1}^z S_{b,1}^z) + J_1(S_{a,0}^z + S_{b,0}^z)\mu_1^z + J_1\mu_1^z(S_{a,1}^z + S_{b,1}^z) - h(S_{a,1}^z + S_{b,1}^z + \mu_1^z)]\}$ we can express $g_n(S_{a,0}, S_{b,0})$ through $g_{n-1}(S_{a,1}, S_{b,1})$

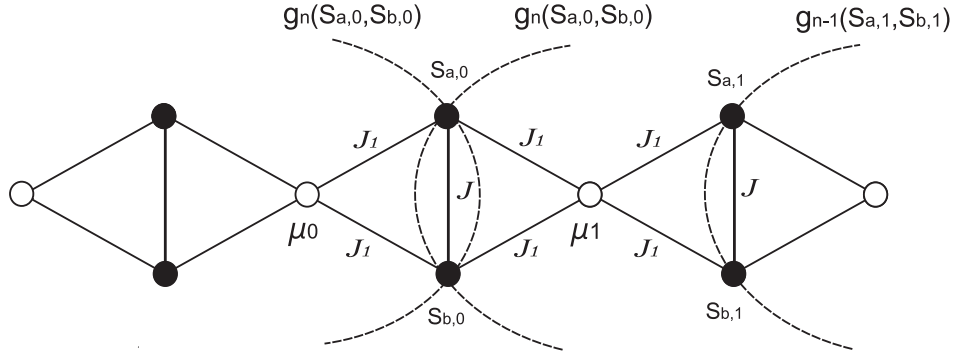


Figure 1: The procedure for derivation of the diamond chain.

$$\begin{aligned}
g_n(\uparrow\uparrow) &= (e^{-\frac{J\Delta}{4T} - \frac{J_1}{T} + \frac{3h}{2T}} + e^{-\frac{J\Delta}{4T} + \frac{J_1}{T} + \frac{h}{2T}})g_{n-1}(\uparrow\uparrow) + (e^{-\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} + \frac{h}{2T}} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} - \frac{h}{2T}})g_{n-1}(\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}) \\
&\quad + (e^{\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} + \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} - \frac{h}{2T}})g_{n-1}(\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}) + e^{-\frac{J\Delta}{4T} - \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{3h}{2T}})g_{n-1}(\downarrow\downarrow), \\
g_n(\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}) &= (e^{-\frac{J\Delta}{4T} - \frac{J_1}{2T} + \frac{3h}{2T}} + e^{-\frac{J\Delta}{4T} + \frac{J_1}{2T} + \frac{h}{2T}})g_{n-1}(\uparrow\uparrow) + (e^{-\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{h}{2T}} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{h}{2T}})g_{n-1}(\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}) \\
&\quad + (e^{\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{h}{2T}})g_{n-1}(\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}) + (e^{-\frac{J\Delta}{4T} + \frac{J_1}{2T} - \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{J_1}{2T} - \frac{3h}{2T}})g_{n-1}(\downarrow\downarrow), \\
g_n(\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}) &= (e^{-\frac{J\Delta}{4T} - \frac{J_1}{2T} + \frac{3h}{2T}} + e^{-\frac{J\Delta}{4T} + \frac{J_1}{2T} + \frac{h}{2T}})g_{n-1}(\uparrow\uparrow) + (e^{-\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{h}{2T}} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{h}{2T}})g_{n-1}(\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}) \\
&\quad + (e^{\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{h}{2T}})g_{n-1}(\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}) + (e^{-\frac{J\Delta}{4T} + \frac{J_1}{2T} - \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{J_1}{2T} - \frac{3h}{2T}})g_{n-1}(\downarrow\downarrow), \\
g_n(\downarrow\downarrow) &= (e^{-\frac{J\Delta}{4T} + \frac{3h}{2T}} + e^{-\frac{J\Delta}{4T} + \frac{h}{2T}})g_{n-1}(\uparrow\uparrow) + (e^{-\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} + \frac{h}{2T}} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} - \frac{h}{2T}})g_{n-1}(\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}) \\
&\quad + (e^{\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} + \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} - \frac{h}{2T}})g_{n-1}(\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}) + (e^{-\frac{J\Delta}{4T} + \frac{J_1}{T} - \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{J_1}{T} - \frac{3h}{2T}})g_{n-1}(\downarrow\downarrow).
\end{aligned} \tag{6}$$

As it can be seen from relations (6) $g_n(\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}) = g_n(\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}})$ hence our recursion relation will be two-dimensional rational mapping. By introducing the following notations

$$\begin{aligned}
x_n &= \frac{g_n(\uparrow\uparrow)}{g_n(\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}})}, \\
y_n &= \frac{g_n(\downarrow\downarrow)}{g_n(\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}})},
\end{aligned} \tag{7}$$

we can get two-dimensional recursion relation for the partition function

$$\begin{aligned}
x_n &= [(e^{-\frac{J\Delta}{4T} - \frac{J_1}{T} + \frac{3h}{2T}} + e^{-\frac{J\Delta}{4T} + \frac{J_1}{T} + \frac{h}{2T}})x_{n-1} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} + \frac{h}{2T}} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} - \frac{h}{2T}} \\
&\quad + e^{\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} + \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} - \frac{h}{2T}} + (e^{-\frac{J\Delta}{4T} - \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{3h}{2T}})y_{n-1}] \\
&\quad / [(e^{-\frac{J\Delta}{4T} - \frac{J_1}{2T} + \frac{3h}{2T}} + e^{-\frac{J\Delta}{4T} + \frac{J_1}{2T} + \frac{h}{2T}})x_{n-1} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{h}{2T}} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{h}{2T}} \\
&\quad + e^{\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{h}{2T}} + (e^{-\frac{J\Delta}{4T} + \frac{J_1}{2T} - \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{J_1}{2T} - \frac{3h}{2T}})y_{n-1}] \\
y_n &= [(e^{-\frac{J\Delta}{4T} + \frac{3h}{2T}} + e^{-\frac{J\Delta}{4T} + \frac{h}{2T}})x_{n-1} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} + \frac{h}{2T}} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} - \frac{h}{2T}} \\
&\quad + e^{\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} + \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} - \frac{h}{2T}} + (e^{-\frac{J\Delta}{4T} + \frac{J_1}{T} - \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{J_1}{T} - \frac{3h}{2T}})y_{n-1}] \\
&\quad / [(e^{-\frac{J\Delta}{4T} - \frac{J_1}{2T} + \frac{3h}{2T}} + e^{-\frac{J\Delta}{4T} + \frac{J_1}{2T} + \frac{h}{2T}})x_{n-1} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{h}{2T}} + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{h}{2T}} \\
&\quad + e^{\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{h}{2T}} + (e^{-\frac{J\Delta}{4T} + \frac{J_1}{2T} - \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{J_1}{2T} - \frac{3h}{2T}})y_{n-1}].
\end{aligned} \tag{8}$$

Recursion relation (8) plays a crucial role in our further investigation because the thermodynamic quantities like magnetization can be expressed through two-dimensional rational mapping. Magnetization for the sublattice of Heisenberg spins can be found using the following formula

$$m_H = \frac{\langle S_{a,i}^z + S_{b,i}^z \rangle}{2} = \frac{\langle S_{a,0}^z + S_{b,0}^z \rangle}{2} \quad (9)$$

$$= \frac{\sum_{\{S_{a,0}, S_{b,0}\}} (S_{a,0}^z + S_{b,0}^z) e^{-\beta[J(S_{a,0}^x S_{b,0}^x + S_{a,0}^y S_{b,0}^y + \Delta S_{a,0}^z S_{b,0}^z) - h(S_{a,0}^z + S_{b,0}^z)]} g_n^2(S_{a,0}, S_{b,0})}{2Z}.$$

In Eq. (9) the sum goes over all possible combinations of $S_{a,0}$ and $S_{b,0}$. Putting into Eq. (9) expression for the partition function and taking into account the notation (7) we can express magnetization for the sublattice of Heisenberg spins through recursion relations which can be written as

$$m_H = \frac{e^{-\frac{J\Delta}{4T} + \frac{h}{T}} x_n^2 - e^{-\frac{J\Delta}{4T} - \frac{h}{T}} y_n^2}{2(e^{-\frac{J\Delta}{4T} + \frac{h}{T}} x_n^2 + e^{-\frac{J}{2T} + \frac{J\Delta}{4T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T}} + e^{-\frac{J\Delta}{4T} - \frac{h}{T}} y_n^2)}. \quad (10)$$

In the same way we can find magnetization for the sublattice of Ising spins.

$$m_I = \langle \mu_i \rangle = \langle \mu_1 \rangle = \frac{\sum_{\{\mu_i, S_{a,0}, S_{b,0}\}} \mu_1 e^{-\beta[J(S_{a,0}^x S_{b,0}^x + S_{a,0}^y S_{b,0}^y + \Delta S_{a,0}^z S_{b,0}^z) - h(S_{a,0}^z + S_{b,0}^z)]} g_n^2(S_{a,0}, S_{b,0})}{Z}. \quad (11)$$

In this expression μ_1 is a part of right branch of $g_n(S_{a,0}, S_{b,0})$. So to find magnetization for the sublattice of Ising spins we need to express $g_n(S_{a,0}, S_{b,0})$ through $g_{n-1}(S_{a,1}, S_{b,1})$. It is important to mention that this procedure also should be done for the partition function. After this procedure the expression for the magnetization of the sublattice of Ising spins can be expressed through recursion relation (8):

$$m_I = [(f_1(x_{n-1}, y_{n-1}) - f_2(x_{n-1}, y_{n-1}))e^{-\frac{J\Delta}{4T} + \frac{h}{T}} x_n + (f_3(x_{n-1}, y_{n-1}) - f_4(x_{n-1}, y_{n-1}))(e^{-\frac{J}{2T} + \frac{J\Delta}{4T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T}}) \quad (12)$$

$$+ (f_5(x_{n-1}, y_{n-1}) - f_6(x_{n-1}, y_{n-1}))e^{-\frac{J\Delta}{4T} - \frac{h}{T}} y_n] / [(f_1(x_{n-1}, y_{n-1}) + f_2(x_{n-1}, y_{n-1}))e^{-\frac{J\Delta}{4T} + \frac{h}{T}} x_n$$

$$+ (f_3(x_{n-1}, y_{n-1}) + f_4(x_{n-1}, y_{n-1}))(e^{-\frac{J}{2T} + \frac{J\Delta}{4T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T}}) + (f_5(x_{n-1}, y_{n-1}) + f_6(x_{n-1}, y_{n-1}))e^{-\frac{J\Delta}{4T} - \frac{h}{T}} y_n],$$

where

$$f_1(x, y) = e^{-\frac{J\Delta}{4T} - \frac{J_1}{T} + \frac{3h}{2T}} x + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} + \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} + \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{h}{2T}} y, \quad (13)$$

$$f_2(x, y) = e^{-\frac{J\Delta}{4T} + \frac{J_1}{T} + \frac{h}{2T}} x + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} - \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} - \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{3h}{2T}} y,$$

$$f_3(x, y) = e^{-\frac{J\Delta}{4T} - \frac{J_1}{2T} + \frac{3h}{2T}} x + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} + \frac{J_1}{2T} - \frac{h}{2T}} y,$$

$$f_4(x, y) = e^{-\frac{J\Delta}{4T} + \frac{J_1}{2T} + \frac{h}{2T}} x + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{J_1}{2T} - \frac{3h}{2T}} y,$$

$$f_5(x, y) = e^{-\frac{J\Delta}{4T} + \frac{3h}{2T}} x + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} + \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} + \frac{J_1}{2T} + \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} + \frac{J_1}{T} - \frac{h}{2T}} y,$$

$$f_6(x, y) = e^{-\frac{J\Delta}{4T} + \frac{h}{2T}} x + e^{-\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} - \frac{h}{2T}} + e^{\frac{J}{2T} + \frac{J\Delta}{4T} - \frac{J_1}{2T} - \frac{h}{2T}} + e^{-\frac{J\Delta}{4T} - \frac{J_1}{T} - \frac{3h}{2T}} y.$$

Expressions (10) and (12) will let us calculate the total single-site magnetization of the spin- $\frac{1}{2}$ Ising-Heisenberg model on diamond chain which can be written as

$$m = \frac{m_I + 2m_H}{3}. \quad (14)$$

Figure 2 shows the field behavior of the total magnetization for antiferromagnetic case at the fixed values of interaction constants $J = 1.5$ and $J_1 = 1$, anisotropy parameter $\Delta = 1$ and different values of the absolute temperature (T). At high temperatures the magnetization curve has a monotone structure (Fig. 2 (a)). At lower temperatures the plateau of magnetization at one third is arising in magnetization curve (Fig. 2 (b)). Other plots of the magnetization curves for the different values of the anisotropy parameter Δ are displayed in Fig. 2 ((c), (d), (e), (f)). Figures 2 (b), (c) and (d) show that the larger positive values of the anisotropy parameter correspond to the larger width of the magnetization plateau for the fixed value of the absolute temperature. While for the negative values of anisotropy parameter magnetization curves remain the same (Fig. 2 (e), (f)). As it can be seen from the figures recursion relation

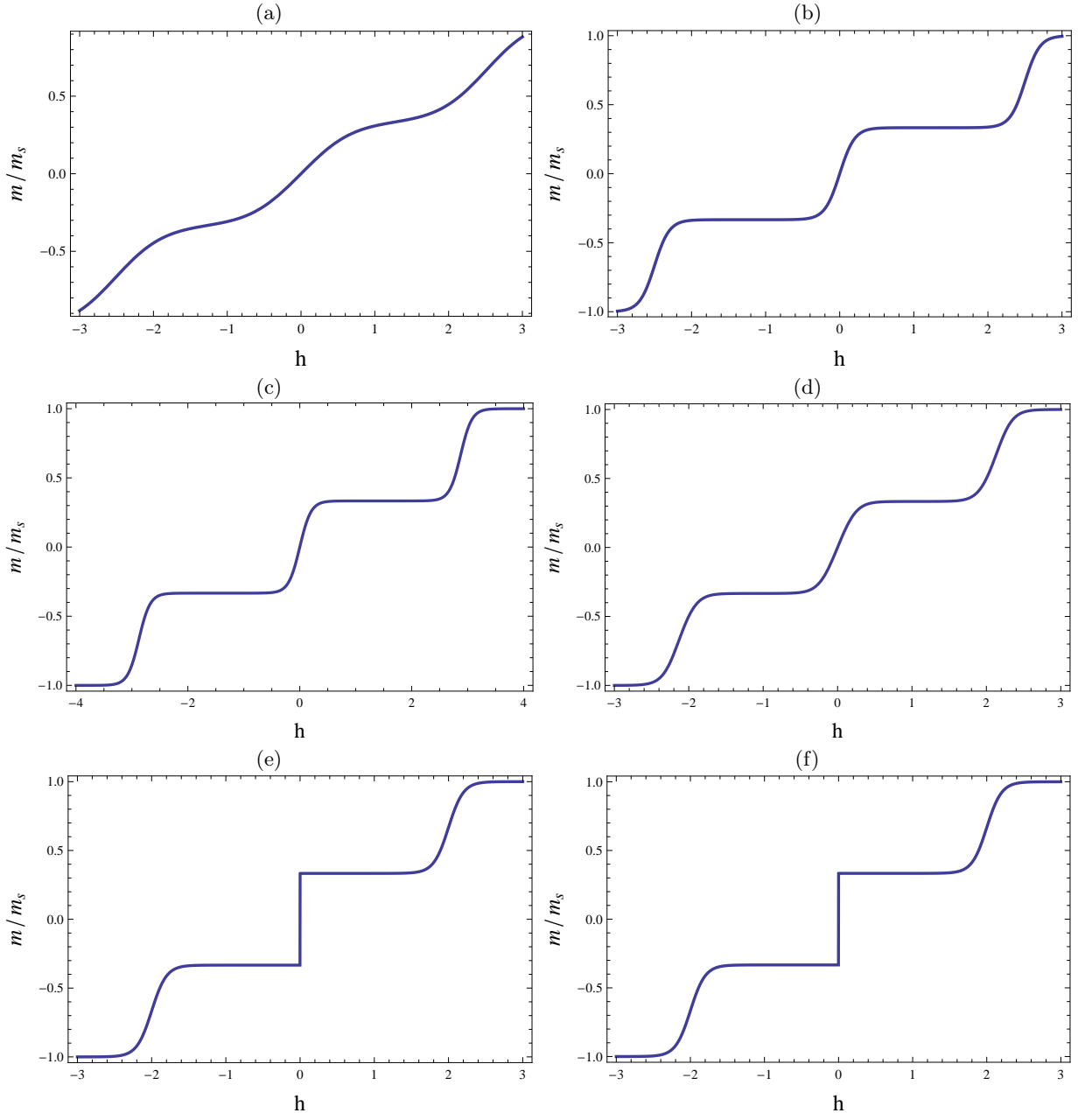


Figure 2: The field dependence of the total magnetization with respect to its saturation value at exchange parameters $J = 1.5$ and $J_1 = 1$: (a) $T = 0.3, \Delta = 1$; (b) $T = 0.1, \Delta = 1$; (c) $T = 0.1, \Delta = 1.5$; (d) $T = 0.1, \Delta = 0.5$; (e) $T = 0.1, \Delta = -1$; (f) $T = 0.1, \Delta = -1.5$.

method results are good agreement with other methods results such as the decoration-iteration transformation method (for example see [4] figure 3).

Let us research the ground state of the spin- $\frac{1}{2}$ Ising-Heisenberg model on diamond chain via Δ and h for the antiferromagnetic ($J = 1.5, J_1 = 1$) and ferromagnetic ($J = -1.5, J_1 = -1$) models. Depending on the value of ratio $\frac{\Delta}{J_1}$ and the magnetic field measured in unites of J_1 , the system exhibits two ferrimagnetic (FRI1 and FRI2) and one paramagnetic (PRM) ground-state phases (Fig. 3(a)) for the antiferromagnetic case. Phases FRI1, FRI2 and PRM

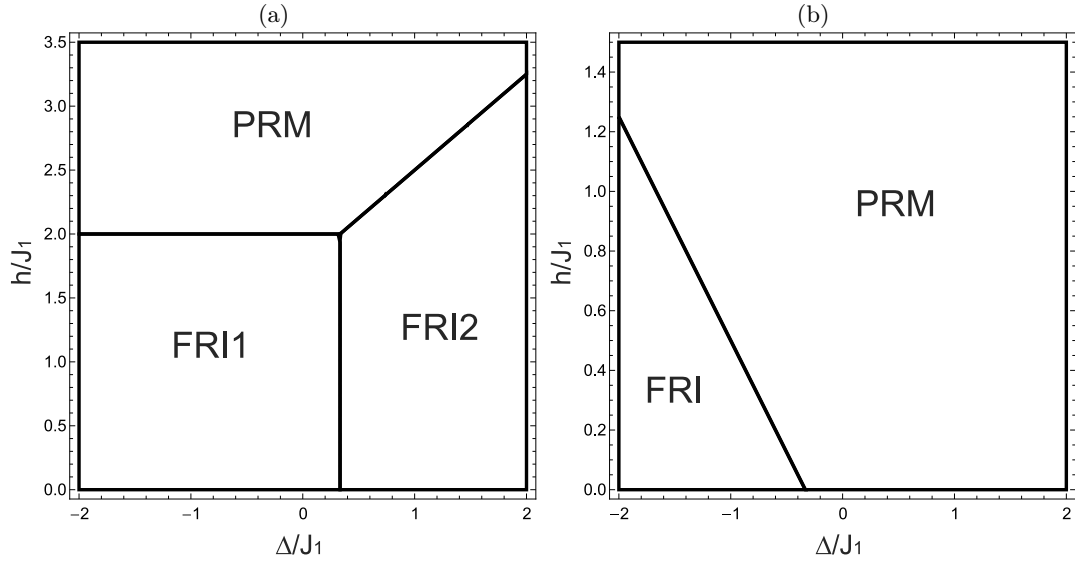


Figure 3: Ground-state phase diagram in the $\Delta - h$ plane for (a) antiferromagnetic case $J = 1.5$, $J_1 = 1$ (b) ferromagnetic case $J = -1.5$, $J_1 = -1$.

correspond to the following values of Ising and Heisenberg spins sublattice magnetization:

$$\begin{aligned} FRI1 : m_I &= -0.5, m_H = 0.5, \\ FRI2 : m_I &= 0.5, m_H = 0, \\ PRM : m_I &= 0.5, m_H = 0.5. \end{aligned} \quad (15)$$

Analytically it can be shown that for the fixed values of exchange parameters $J = 1.5$ and $J_1 = 1$ phase transition from FRI1 to FRI2 takes place at $\Delta = \frac{1}{3}$. Now let us compare the displayed magnetization curves (Fig. 2) with the ground-state phase diagram shown in Fig. 3(a). As it is already mentioned in FRI2 phase the larger positive values of the anisotropy parameter ($\Delta > \frac{1}{3}$) correspond to the larger width of the magnetization plateau see Fig. 3 (b), (c) and (d). In FRI1 phase for the fixed values of interaction constants and the absolute temperature the behavior of the magnetization curve remains the same (Fig. 2 (e), (f)).

For the ferromagnetic case there are two phases in the phase diagram; the ferrimagnetic (FRI) and paramagnetic (PRM) (Fig. 3(b)). Phases FRI and PRM correspond to the following values of Ising and Heisenberg spins sublattice magnetization:

$$\begin{aligned} FRI : m_I &= 0.5, m_H = 0, \\ PRM : m_I &= 0.5, m_H = 0.5. \end{aligned} \quad (16)$$

It can be analytically shown that FRI phase ends on $\Delta = -\frac{1}{3}$ at absence of an external magnetic field.

III. LYAPUNOV EXPONENT AND SUPERSTABLE POINT

In this section we will focus on the thermodynamical equilibrium description of the spin- $\frac{1}{2}$ Ising- Heisenberg model on a diamond chain, by studying infinite-size systems. Lyapunov exponents near the magnetization plateau of the antiferromagnetic model are interesting to calculate on a diamond chain. It is shown that the behavior of the maximal Lyapunov exponent via magnetic field of multi-dimensional rational mapping has a plateau and coincides with magnetization one on one-dimensional kagome chain at low temperatures [35]. It was obtained that the maximal Lyapunov exponent had a negative vanishing plateau.

The following values of Lyapunov exponent can be observed during the investigation.

1. $\lambda < 0$. Negative Lyapunov exponents show that the system is dissipative or non-conservative. The systems with more negative values of Lyapunov exponent are more stabile. If $\lambda = -\infty$ means that we have superstable fixed and superstable periodic points.

2. $\lambda = 0$ corresponding to neutral fixed point. Zero values of Lyapunov exponents are characteristic for conservative systems. At this value of Lyapunov exponents the second-order phase transition takes place.

3. $\lambda > 0$ corresponding to unstable and chaotic systems. The systems with positive Lyapunov exponents have chaotic behavior.

In general for the mapping $x_n = f(x_{n-1})$ Lyapunov exponent $\lambda(x)$ characterizes the exponential divergence of two nearby points after n iterations. Lyapunov exponent may be expressed as a limit of mapping stability as [28–34]

$$\lambda(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \frac{df^n(x)}{dx} \right|. \quad (17)$$

In multidimensional case with dimension n , exists n of exponents for various directions in space

$$e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n} = \lim_{n \rightarrow \infty} (\text{eigenvalues of the product } \prod_{i=0}^{n-1} J(\vec{x}_i))^{\frac{1}{n}}, \quad (18)$$

where $J(\vec{x}) = (\frac{\partial G_i}{\partial x_j})$ is the Jacobian of the mapping $\vec{x}_{n+1} = G(\vec{x}_n)$. For two dimensional mapping (8) we can receive the following expression of Lyapunov exponents

$$\lambda_1, \lambda_2 = \lim_{n \rightarrow \infty} \frac{1}{n} \ln(\text{eigenvalues of the product } \prod_{i=0}^{n-1} J(x_i, y_i)) \quad (19)$$

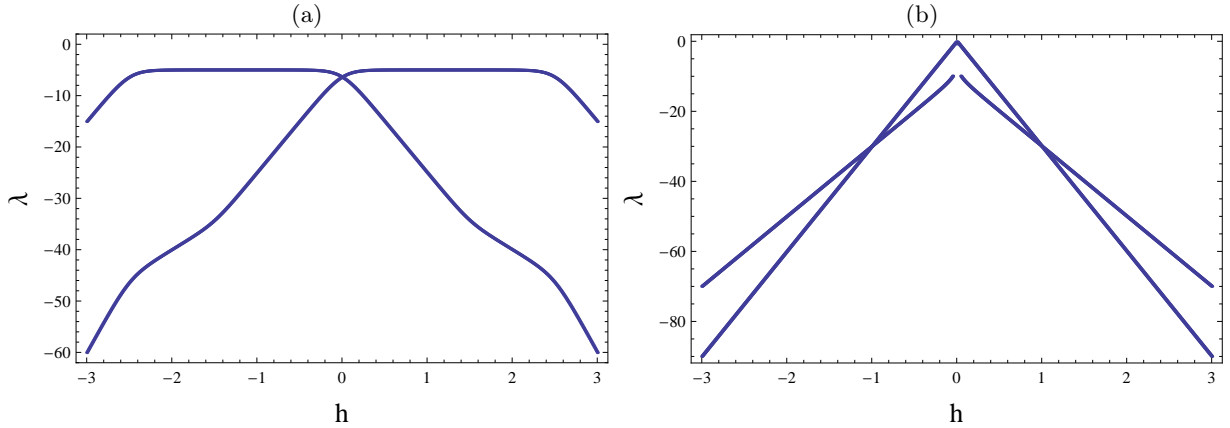


Figure 4: Plot of Lyapunov exponents for spin $\frac{1}{2}$ Ising-Heisenberg model on diamond chain at exchange parameters (a) the antiferromagnetic case at $J = 1.5$, $J_1 = 1$, $\Delta = 1$ and temperature $T = 0.1$ (b) the ferromagnetic case at $J = -1.5$, $J_1 = -1$, $\Delta = 1$ and temperature $T = 0.1$.

where $J(x, y)$ is the Jacobian of the mapping (8). Expression (19) will let us count up the meanings of Lyapunov exponents depending from an external magnetic field (h) at fixed values of constants of interaction (J, J_1), the anisotropy parameter (Δ) and temperature (T). Figure 4 shows the dependence of Lyapunov exponents on an external magnetic field for the antiferromagnetic and ferromagnetic cases. As it can be seen from Fig. 4 (a) both values of Lyapunov exponent are equal to each other at absence of an external magnetic field. Another interesting property of Lyapunov exponent is the existence of the plateau on maximal Lyapunov exponent curve for the antiferromagnetic case. It is important to mention that the locations of plateaus on the maximal Lyapunov exponent curve (Fig. 4 (a)) coincide with the locations of plateaus on the magnetization curve (Fig. 2 (b)). Such an interesting phenomena of Lyapunov exponent has also been observed by investigating two, three and six spin exchange interactions Heisenberg model on kagome lattice in an external magnetic field [35]. Figure 4 (b) shows that for the ferromagnetic case the maximum value of the maximal Lyapunov exponent tends to zero.

In Fig. 5 we show the behavior of Lyapunov exponent for the antiferromagnetic case at lower temperature. As it can be seen from figure the absolute values of Lyapunov exponents are increasing by decreasing the temperature.

Next, we turn our attention to the behavior of Lyapunov exponent curves in different ferrimagnetic (FRI1 and FRI2) phases. For this purpose, Lyapunov exponent curves for fixed values of interaction constants and the absolute temperature are plotted in Fig. (6). At low temperatures, the minimum and maximum Lyapunov exponents have

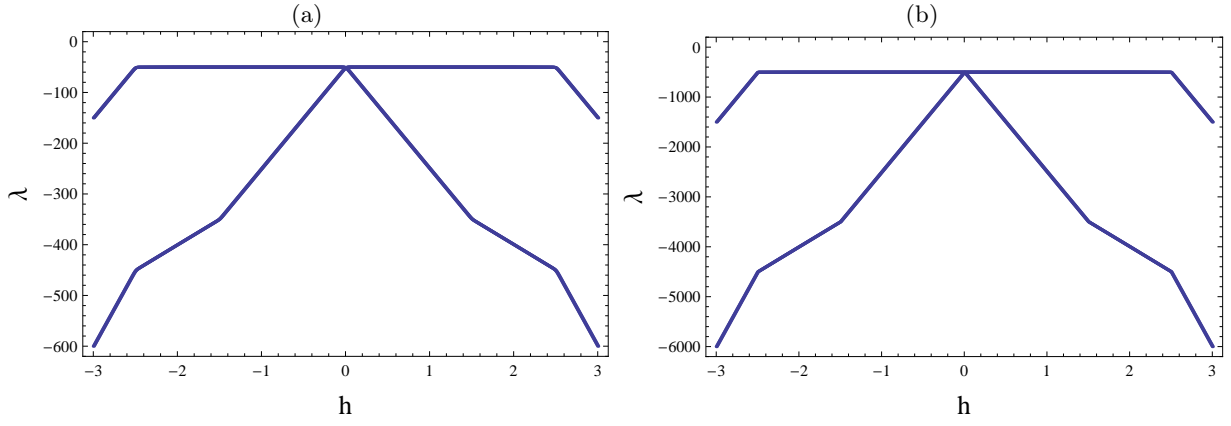


Figure 5: Plot of Lyapunov exponents at exchange parameters $J = 1.5$, $J_1 = 1$, $\Delta = 1$ and temperature (a) $T = 0.01$ (b) $T = 0.001$.

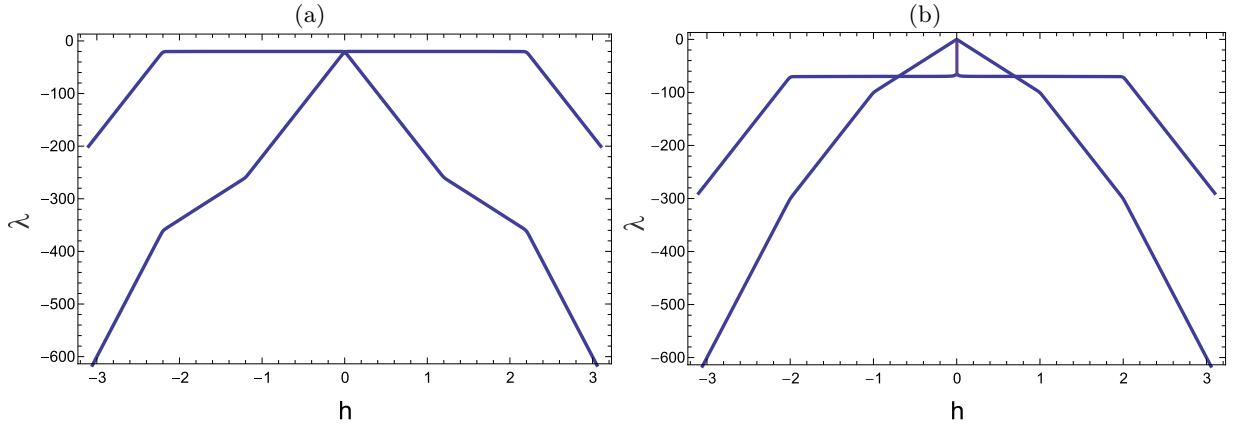


Figure 6: Plot of Lyapunov exponents for different ferrimagnetic (FRI1 and FRI2) phases at exchange parameters $J = 1.5$, $J_1 = 1$ and the temperature $T = 0.01$ (a) $\Delta = 0.6$ (b) $\Delta = -0.6$.

different behavior. Only at $h = 0$ the maximum value of the minimum Lyapunov exponent equal to value of the maximum one, when the system is in FRI2 phase (Fig. 6 (a)), and there is no intersection of Lyapunov exponents for $h \neq 0$. There is a super stable point ($\lambda_{max} \rightarrow \infty$), when $h = 0$ and at $T \rightarrow \infty$ in FRI2 phase. There are two points of intersections for the maximum and minimum values of Lyapunov exponents and coincide when $h = 0$ in FRI1 phase (Fig. 6 (b)). Lyapunov exponents are tending to zero in thermodynamic limit ($T \rightarrow 0$) in FRI1 phase at absence of an external magnetic field.

Now let us investigate another interesting property of the recursion relation (8), namely superstability. First of all we will define superstability for the one dimensional recursion relation. Generally one dimensional recursion relation $x_n = f(x_{n-1})$ is said to be superstable if the following relation takes place [36–39]

$$\frac{d^n f(x^*)}{dx} = 0, \quad (20)$$

where x^* is the fixed point of $f(x)$. An other way superstability can be defined by using definition of Lyapunov exponent. The system is superstable when

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \frac{d^n f(x^*)}{dx} \right| = -\infty. \quad (21)$$

In the same way we can define superstability for two dimensional recursion relations (8). As it is already mentioned above for the antiferromagnetic case the absolute values of Lyapunov exponents are increasing by decreasing the temperature. Putting values of the exchange parameters ($J = 1.5$, $J_1 = 1$ and $\Delta = 1$) into equation (17) we can see

that at thermodynamic limit at absence of an external magnetic field the following relation takes place for Lyapunov exponents

$$\lim_{T \rightarrow 0} \lambda_1 = \lambda_2 = -\infty, \quad (22)$$

which shows the existence of the super stable point.

We have analyzed the behavior of the magnetization for the spin- $\frac{1}{2}$ Ising-Heisenberg model on diamond chain for different values of anisotropy parameter Δ . For the antiferromagnetic case ($J > 0$, $J_1 > 0$) at fixed values of exchange parameters $J = 1.5$ and $J_1 = 1$, the temperature T and for $\Delta < \frac{1}{3}$ the magnetization curves have the same appearance as in Fig. 2 (e). The values of characteristic Lyapunov exponent tend to zero at absence of an external magnetic field and at $T \rightarrow 0$, which means that there is no supercritical behavior for the antiferromagnetic case when $\Delta < \frac{1}{3}$.

For the antiferromagnetic case ($J > 0$, $J_1 > 0$) at low temperatures and for values of the anisotropy parameter Δ ($\Delta > \frac{1}{3}$) the magnetization curves have the same behavior as shown in Fig. 2 (b). The changes of the anisotropy parameter Δ only brings to the changes of the width of the magnetization plateau at one third. The values of characteristic Lyapunov exponent tend to $-\infty$ at absence of an external magnetic field and at $T \rightarrow 0$, which means that there is a superstable point for the antiferromagnetic case for positive values of anisotropy parameter Δ . Usually a super stable point lies between bifurcation points [36–39]. In our case for the spin- $\frac{1}{2}$ Ising-Heisenberg model on a diamond chain there are no bifurcation points but the maximal Lyapunov exponent tends to minus infinity. So we get the phase transition in the super stable point at $h = 0$ and $T \rightarrow 0$. For the first time we get the phase transition point at the super stable one.

IV. CONCLUSION

By using the recursion relation technique, we have studied magnetic properties of the exactly solvable spin- $\frac{1}{2}$ Ising-Heisenberg model on diamond chain. Recursion relation technique allowed us to construct the exact two-dimensional recursion relation for the partition function. The behavior of the total magnetization with respect to its saturation value has been investigated. The existence of the magnetization plateau at one third of saturation value of magnetization has been observed in the antiferromagnetic case. The ground-state phase diagrams in $\Delta - h$ plane show the existence of two ferrimagnetic (FRI1 and FRI2) phases and one paramagnetic (PRM) phase in the antiferromagnetic case and one ferrimagnetic (FRI) and a paramagnetic (PRM) phases in the ferromagnetic case.

The properties of Lyapunov exponents were also discussed. The existence of the plateau of the maximal Lyapunov exponent curve was observed at low temperatures. It was detected the different behavior for Lyapunov exponent curves in two ferrimagnetic phases. We have shown that for the antiferromagnetic case in the thermodynamic limit ($T \rightarrow 0$) both values of Lyapunov exponent tend to $-\infty$ at absence of an external magnetic field which is corresponding to the superstable point.

V. ACKNOWLEDGMENTS

This work has been supported by the French-Armenian Grant No. CNRS IE-017 and Marie Curie IRSES SPIDER, and project PIRSES-GA-2011-295302 (NA). The authors are grateful to R. Kenna and H. Lazaryan for useful discussions.

-
- [1] H.-J. Mikeska, C. Luckmann, arXiv:0709.2863.
 - [2] O. Derzhko and J. Richter, Eur. Phys. J. B 52 (2006) 23;
O. Derzhko, J. Richter, A. Honecker, H.-J. Schmidt, Low Temp. Phys. 33 (2007) 745.
 - [3] K. Okamoto, T. Tonegawa, M. Kaburagi, J. Phys.: Condens. Matter 15 (2003) 5979.
 - [4] L. Čanová, J. Strečka, M. Jaščur, J. Phys.: Condens. Matter 18 (2006) 4967.
 - [5] B. M. Lisnii, Low Temp. Phys. 37 (2011) 296.
 - [6] M. S. S. Pereira, F. A. B. F. de Moura, M. L. Lyra, Phys. Rev. B 77 (2008) 024402;
M. S. S. Pereira, F. A. B. F. de Moura, M. L. Lyra, Phys. Rev. B 79 (2009) 054427.
 - [7] O. Rojas, S. M. de Souza, Phys. Lett. A 375 (2011) 1295.
 - [8] J. Strečka, M. Jaščur, J. Phys.: Condens. Matter 15 (2003) 4519.
 - [9] B. Gu, G. Su, Phys. Rev. B 75 (2007) 174437.
 - [10] N. Ananikian, H. Lazaryan, M. Nalbandyan, Eur. Phys. J. B 85 (2012) 223.

- [11] N. S. Ananikian, L. N. Ananikyan, L. A. Chakhmakhchyan, O. Rojas, J. Phys.: Condens. Matter 24 (2012) 256001.
- [12] J. Kang, C. Lee, R. K. Kremer, M.-H. Whangbo, J. Phys. Condens. Matter 21 (2009) 392201.
- [13] H. Jeschke et al., Phys. Rev. Lett. 106 (2011) 217201.
- [14] H. Kikuchi, Y. Fujii, M. Chiba, S. Mitsudo, T. Idehara, Physica B 329-333 (2003) 967;
H. Kikuchi et al., Phys. Rev. Lett. 94 (2005) 227201;
H. Kikuchi et al., Prog. Theor. Phys. Suppl. 159 (2005) 1.
- [15] N. S. Ananikian, V. V. Hovhannisyan, H. A. Lazaryan, Int. J. Mod. Phys. B 24 (2010) 5913.
- [16] N. S. Ananikian, L. N. Ananikyan, R. Artuso, V. V. Hovhannisyan, Physica D 239 (2010) 1723.
- [17] V. V. Hovhannisyan, L. N. Ananikyan, N. S. Ananikian, Int. J. of Mod. Phys. B 21 (2007) 3567.
- [18] T. A. Arakelyan et al., Phys. Rev. B. 67 (2003) 024424.
- [19] C.-K. Hu, N. Sh. Izmailian, K. B. Oganessian, Phys. Rev. E 59 (1999) 6489.
- [20] E. Albayrak, O. Canko, Physica A 373 (2007) 363;
E. Albayrak, S. Yilmaz, Physica A 387 (2008) 1173;
E. Albayrak, S. Akkaya, T. Cengiz, J. Magn. Magn. Mat. 321 (2009) 108.
- [21] J. Strečka, C. Ekiz, Physica A 391 (2012) 4763;
C. Ekiz, J. Strečka, M. Jascur, Cent. Eur. J. Phys. 7 (2009) 509.
- [22] E. Albayrak, A. Yigit, Phys. Lett. A 353 (2006) 121;
E. Albayrak, Physica A 375 (2007) 174.
- [23] N. Ananikian, L. Ananikyan, R. Artuso, Phys Lett. A 4 (2007) 615.
- [24] A. Z. Akheyany, N. S. Ananikian, J. Phys. A: Math. Gen. 29 (1996) 721.
- [25] N. S. Ananikian et al., Physica A 172 (1991) 391.
- [26] A. R. Avakian, N. S. Ananikian, N. Sh. Izmailyan, Phys. Lett. A 150 (1990) 163.
- [27] N. S. Ananikian et al., JETP Lett. 59 (1994) 71.
- [28] Schuster, H.G., *Deterministic Chaos*, Weinheim, 1984.
- [29] V. I. Oseledec, Trans. Moskow Math. Soc. 19 (1968) 197.
- [30] J.-P. Eckmann, D. Ruelle, Rev. Mod. Phys. 57 (1985) 617.
- [31] A. Crisanti, G. Paladin, A. Vulpiani, *Products of Random Matrices in Statistical Physics*, Springer, Berlin, 1993.
- [32] V. Latora, A. Rapisarda, S. Ruffo, Phys. Rev. Lett. 80, 692 (1998);
V. Latora, A. Rapisarda, S. Ruffo, Physica A 280 (2000) 81.
- [33] Í. Birol, A. Hacinliyan, Phys. Rev. E 52 (1995) 4750.
- [34] M. Keskin, O. Canko, B. Deviren, Phys. Rev. E 74 (2006) 011110.
- [35] A. Ananikian, L. Ananikyan, R. Artuso, H. Lazaryan, Phys. Lett. A 374 (2010) 4084.
- [36] K. Kaneko, Prog. Theor. Phys. Suppl. 72 (1984) 1089.
- [37] M. Howard Lee, J. Math. Phys. 50 (2009) 122702.
- [38] E. Ott, *Chaos in Dynamical Systems*, Cambridge University Press 1993.
- [39] N. S. Ananikian, L. N. Ananikyan, L. A. Chakhmakhchyan, JETP L. 94 (2011) 39.